RESEARCH DEPARTMENT

THE DISTRIBUTION OF CURRENT ON CYLINDRICAL RADIATING DIPOLES

Report No. E-078

(1962/24)

THE BRITISH BROADCASTING CORPORATION
ENGINEERING DIVISION

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SUMMARY

The distribution of current on a dipole comprising cylindrical conductors is approximately sinusoidal, but in certain problems a more accurate solution is required. A number of theoretical attempts have been made to obtain a better approximation and are in reasonable agreement. These methods are, however, laborious to apply in practice and are also limited in application.

In this report a simple method is described which enables a current distribution more accurate than the sinusoidal approximation to be determined. The method is based on the perturbation principle and besides being applicable to a wide variety of cases, is in good agreement with more rigorous theoretical approaches and with measurements.

1. INTRODUCTION

A radiating dipole comprising cylindrical conductors behaves approximately as an open-circuited uniform lossless balanced transmission line as regards the distribution of current along its length. The current is thus distributed approximately sinusoidally and the radiation pattern can be calculated on this basis. For many applications, the assumption of a sinusoidally distributed current results in a calculated radiation pattern which is sufficiently accurate for practical purposes. In certain cases, however, an assumption of a sinusoidally distributed current can lead to appreciable error in the radiation pattern and a better approximation is needed. The calculation of a more accurate distribution is, however, a matter of some difficulty since it is not possible to state the field conditions at the tips of the dipole precisely.

It is convenient to assume that the true dipole current is the sum of a sinusoidal primary current (a standing wave pattern with a velocity of propagation equal to that of light) and a correcting term which may be called the secondary current.[1] The secondary current may be resolved into two components, one in phase and one in quadrature with the primary current. The primary current and the in-phase component of the secondary current may be regarded as a modified primary current and result in a standing wave pattern which is substantially sinusoidal but with a velocity of propagation less than that of light. The quadrature component of the secondary current can conveniently be called the "feed" current[2] since it is in phase with the driving voltage and is associated with the radiation of power.

Theoretical attempts have been made[2-8] to obtain an approximation to the secondary current and are in qualitative agreement on the manner in which the true current distribution varies from a sinusoidal distribution. However, these methods are laborious to apply to practical cases and difficulties arise in the case of a dipole which is mutually coupled to other aerials.

In this report a method based on the perturbation principle is described which enables a current distribution more accurate than the sinusoidal distribution to be determined. The method is simple to apply and gives a result for the "feed" current which is in good agreement with that derived from more rigorous approaches. In addition, the solution of cases in which the aerial is mutually coupled to other aerials is straightforward. The extension of the method to predict the in-phase component of the secondary current is not simple, but this is not thought to be a disadvantage. The theoretical analyses[2-8] are not in any case in good agreement as regards prediction of the in-phase component and reliable measured figures of the reduction of velocity of propagation are available.[1]

2. BASIS OF METHOD

2.1. The Perturbation Principle

It has been pointed out[9] that the compensation theorem for electrical networks, though capable of exact statement, is principally useful as a perturbation method. Such a method enables the current flowing in any conductor in a linear reciprocal network to be determined from the change of input impedance which occurs when a small perturbing impedance is placed in the path of the current. In many cases it is much simpler to determine the change of input impedance than to calculate the current directly and the method is thus of considerable interest.

If I is the current flowing in a conductor in a network and an impedance δz is inserted in series with the conductor, the input impedance will change by an amount δZ (say). It can be shown that:[9]

$$\delta Z = \frac{I}{I_0^2}(I + \delta I)\ \delta z \qquad (1)$$

where δI is the change of current resulting from the insertion of the perturbing impedance δz and I_0 is the current applied to the input terminals of the network. Equation (1) is exact but provided that δz is made sufficiently small it can be simplified to:[9]

$$\frac{I^2}{I_0^2} = \frac{\delta Z}{\delta z} \qquad (2)$$

Thus the current flowing in any part of a network can be determined from the change of input impedance due to a perturbing impedance. For the purposes of analysis it is usually convenient to assume that I_0 is unity.

2.2. Perturbation of a Cylindrical Dipole

The perturbation method described in Section 2.1 gives a means of obtaining an approximation to the secondary current on a cylindrical dipole; for convenience, however, the electrically identical case of a unipole above a perfectly conducting ground plane will be considered. If the unipole is broken at any point and a small impedance is inserted in series (for convenience a small inductance will be assumed), then the current at the perturbed point can be determined from the change of input

impedance. As an initial approximation the current distribution is assumed to be sinusoidal and this enables an approximation to the input impedance to be readily calculated. This input impedance will be:

$$Z = R + jX \tag{3}$$

where R is the radiation resistance (determined by integration of power over a large sphere) referred to the input terminals and X is the input reactance $-Z_0 \cot 2\pi l/\lambda$ the theoretical value for a lossless transmission line. Z_0 is the characteristic impedance of the unipole* and l the length.

When the perturbing impedance is inserted the current distribution is changed. If this change of current distribution is calculated on the assumption that the unipole behaves as a lossless transmission line the change of input impedance will be:

$$\delta Z = \delta R + j\delta X \tag{4}$$

where δR is the change of radiation resistance due to the change of current distribution and δX is the change of input reactance of a loss-free transmission line. Thus from the change of impedance δZ given by equation (4), an approximation to the current in the unipole at the point of perturbation can be derived from equation (2). This current will differ from the original sinusoidal assumption since account has now been taken of the radiated field by including the change of radiation resistance.

The error in the determination of the secondary current will clearly depend on the accuracy of the initial assumption of sinusoidal current distribution. Thus considerable error will result if the perturbation method is applied to regions of the unipole in which the sinusoidal primary current is small; for example, at or near the drive point of a unipole $\lambda/2$ high. At the top of a unipole both the primary and secondary currents must of necessity tend to zero and the error in the secondary current determined is difficult to assess. The current at the top of a unipole cannot, however, be determined with accuracy by any method due to the difficulties involved in assessing the field in the neighbourhood of the tip.

3. CURRENT DISTRIBUTION OF UNIPOLES OF HEIGHT UP TO $\lambda/4$

The case considered is shown in Fig. 1(a), the unipole having a characteristic impedance of Z_0. The height is h and $2\pi h/\lambda = \phi_0$ where λ is the free space wavelength. Fig. 1(b) shows the equivalent lossless transmission line from which the preliminary assumptions of current distribution before and after perturbation are determined; it is assumed that unit current is injected at the base from a constant current generator.

*Unlike a non-radiating transmission line a unipole cannot have a characteristic impedance since it is not uniform. However, for practical purposes the unipole behaves approximately as a uniform transmission line and a parameter corresponding to the characteristic impedance can be used. Of several formulae, the one in best accord with experimental results is due to G.W.O. Howe[10] and is $Z_0 = 60(\log_e(l/\alpha - 1)$ ohms where l is the length and α the radius of the cylinder.

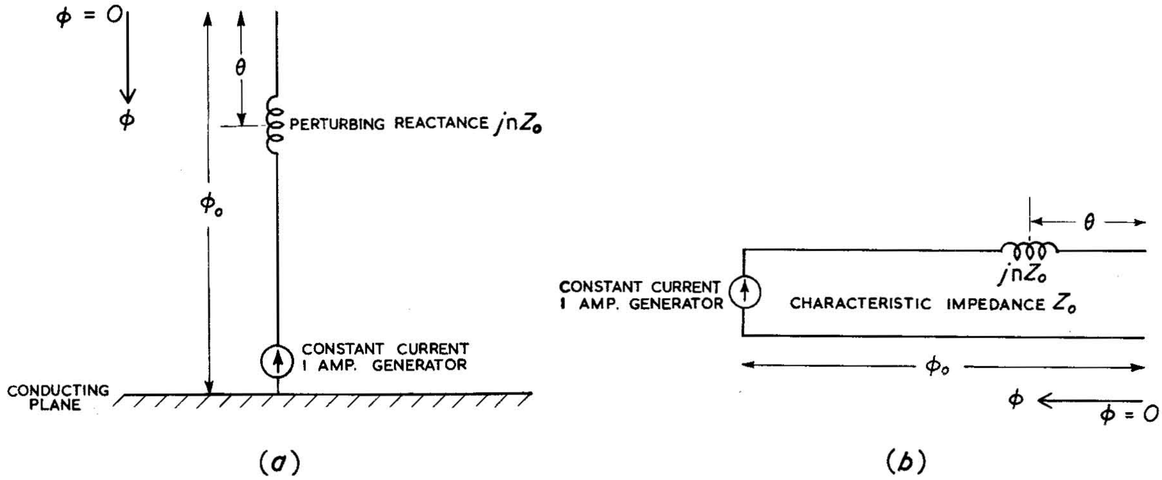

Fig. 1 - Perturbation of isolated unipole

The unipole is perturbed at a distance of θ radians from the top by a small series impedance jnZ_0 and on the assumption of a lossless transmission line the change of input reactance (equation (4)) can be shown to be:

$$j\delta X = jZ_0 n \frac{\sin^2\theta}{\sin^2\phi_0} \tag{5}$$

Before the perturbation is inserted the current at a point ϕ radians from the top of the unipole will be:

$$I_p(\phi) = \frac{\sin\phi}{\sin\phi_0} \tag{6}$$

where the subscript P denotes the sinusoidal primary current. After insertion of the perturbation the current at a point ϕ radians from the top of the unipole will be:

$$\begin{aligned} I_p(\phi) &= \frac{\sin(\theta + \Delta)}{\sin(\phi_0 + \Delta)\sin\theta} \cdot \sin\phi \qquad && 0 \leqslant \phi \leqslant \theta \\ I_p(\phi) &= \frac{\sin(\phi + \Delta)}{\sin(\phi_0 + \Delta)} && \theta \leqslant \phi \leqslant \phi_0 \end{aligned} \tag{7}$$

where $\Delta = n\sin^2\theta$ the effective change of length of the unipole in radians.

If the height of the unipole is sufficiently small, the vertical radiation pattern (v.r.p.) will be unchanged when the perturbation is added. The horizontal radiation pattern (h.r.p.) will be circular irrespective of the perturbation and the change in radiation resistance can thus be determined from the total current moments of the unperturbed and perturbed aerials. These current moments will be M_1 for the unperturbed unipole and M_2 for the perturbed unipole and will be determined by integration of equations (6) and (7) giving:

$$M_1 = \frac{1}{\sin\phi_0}\int_0^{\phi_0} \sin\phi \, d\phi = \tan\frac{\phi_0}{2}$$

$$M_2 = \frac{\sin(\theta + \Delta)}{\sin(\phi_0 + \Delta)\sin\theta}\int_0^{\theta}\sin\phi\, d\phi + \frac{1}{\sin(\phi_0 + \Delta)}\int_{\theta}^{\phi_0}\sin(\phi + \Delta)\, d\phi \tag{8}$$

$$= \tan\frac{\phi_0}{2}\left[1 + \frac{\Delta(\tan\frac{\phi_0}{2} - \tan\frac{\theta}{2})}{1 - \cos\phi_0}\right]$$

Thus if the radiation resistance of the unperturbed unipole (measured as a series component at the drive point) is R_R the radiation resistance of a perturbed unipole is given by:

$$R'_R = \frac{M_2^2}{M_1^2} R_R \tag{9}$$

Substituting for M_1, M_2 and Δ from equations (7) and (8) the change of input resistance due to the perturbation is given by $\delta R = R'_R - R_R$ and is:

$$\delta R = \frac{n \sin^2\theta(\tan\frac{\phi_0}{2} - \tan\frac{\theta}{2})}{\sin^2\frac{\phi_0}{2}} R_R \tag{10}$$

Equations (5) and (10) give the change of input impedance due to the perturbation, and from equation (2) the square of the current at the point of perturbation is given by dividing the change of impedance by the value of the perturbation (jnZ_0); thus:

$$I^2(\theta) = \frac{\delta Z}{\delta z} = \frac{\delta R + j\delta X}{jnZ_0} = \sin^2\theta\left\{\operatorname{cosec}^2\phi_0 - j\frac{R_R}{Z_0}\operatorname{cosec}^2\frac{\phi_0}{2}\left(\tan\frac{\phi_0}{2} - \tan\frac{\theta}{2}\right)\right\}$$

where the absence of the subscript P which was used in equation (7) denotes that the current is now the total current comprising both the primary and secondary terms. It should be noted that the primary currents of equations (6) and (7) and the total current of equation (11) appear as functions of two different variables. The primary current which is a function of ϕ must be integrated over the length of the unipole to find the total current at the perturbed point θ radians from the top of the aerial.

In most practical cases, the imaginary term in equation (11) has a modulus small compared with unity and the current may be determined by approximating to the square root giving:

$$I(\theta) = \sin\theta\left\{\operatorname{cosec}\phi_0 - j\frac{R_R}{Z_0}\left(1 - \tan\frac{\theta}{2}\cot\frac{\phi_0}{2}\right)\right\} \tag{12}$$

where the real term is the sinusoidal "primary" current and the imaginary term is the quadrature "feed" current. The modified current distribution must account for the power radiated from the unipole; the sinusoidal primary current fails to do this since it results in a purely reactive input impedance. The perturbation method assumes that the unipole behaves as a uniform transmission line and the voltage at a point θ from the top is therefore $V(\theta) = -jZ_0\, \partial I(\theta)/\partial\theta$. Applying this relationship

to equation (12), the voltage at $\theta = \phi_0$ (i.e. at the drive point) is $R_R - jZ_0\cot\phi_0$. Since an input current of unity has been assumed the input impedance is $R_R - jZ_0\cot\phi_0$ and the correct radiated power is therefore drawn from the generator.

The relative and not the absolute amplitudes of the two terms corresponding to the dipole current are of interest and it is more convenient to write the sinusoidal term without a multiplying constant. Thus equation (12) becomes:

$$I_r(\theta) = \sin\theta\left(1 - j\frac{R_R}{Z_0}\left(1 - \frac{\tan\frac{\theta}{2}}{\tan\frac{\phi_0}{2}}\right)\sin\phi_0\right) \tag{13}$$

where r denotes the relative current.

As has already been pointed out, equation (13) is based on the assumption that when the perturbation is inserted in the dipole, the change in the shape of the v.r.p. can be neglected. In the Appendix it is shown that this assumption is valid for unipole heights of up to $\lambda/4$ and in this case $\phi_0 = \pi/2$ and equation (13) becomes:

$$I(\theta) = \sin\theta\left[1 - j\frac{R_R}{Z_0}\left(1 - \tan\frac{\theta}{2}\right)\right] \tag{14}$$

Another case of interest is that of a very short unipole with $\phi_0 << \pi/2$ in which case equation (13) becomes:

$$I_r(\theta) = \theta\left\{1 - j\frac{R_R}{Z_0}(\phi_0 - \theta)\right\} \tag{15}$$

In applying equations (13), (14) and (15) in practical cases, values of R_R, the radiation resistance of the unipole referred to the drive point are necessary. Figures for the radiation resistance based on a sinusoidal current distribution for dipoles of arbitrary length are readily available.[11]

4. CURRENT DISTRIBUTION IN THE PRESENCE OF EXTERNAL FIELDS

In Section 3 the current distribution on an isolated unipole was analysed. The case now considered is that of a unipole in the presence of a second aerial of arbitrary type illustrated diagrammatically in Fig. 2. Both aerials are driven from a common source delivering unit current, a current of I_1 being impressed at the unipole input terminals and I_2 at the terminals of the second aerial. An equivalent circuit of the arrangement can be drawn as in Fig. 3 where Z_1 and Z_2 are the self impedances of the unipole and second aerial respectively and Z_m is the mutual impedance.*

It is required to find the current on the unipole and a series perturbation of jnZ_0 is therefore inserted as in the analysis of Section 3. The insertion of the perturbation will cause both the self and mutual impedances of the unipole to

*Z_1 and Z_2 are the impedances of each aerial with the terminals of the other aerial open circuited and Z_m is the open circuit voltage at the terminals of one aerial with 1 ampere injected into the other. The second aerial is assumed to be such that open circuiting its terminals will result in its effect on the unipole being negligible.

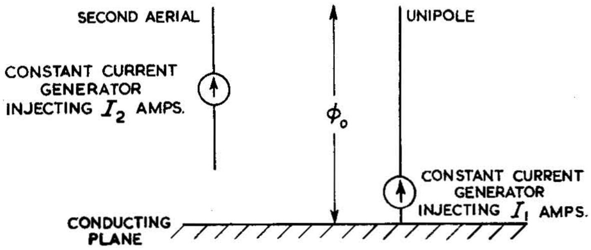

Fig. 2 - Unipole mutually coupled to second aerial

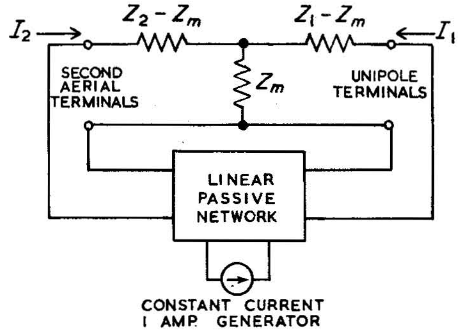

Fig. 3 - Equivalent circuit for analysis of coupled aerials

change; suppose these changes are δZ_1 and δZ_m respectively. The equivalent circuit then becomes as in Fig. 4 and from the theorem of equation (2) the change of input impedance seen by the common source will be:

$$\delta Z = I_1^2 \delta Z_1 + 2 I_1 I_2 \, \delta Z_m \tag{16}$$

Also from the theorem of equation (2) the square of the current on the unipole at the point of perturbation must be:

$$I_1^2(\theta) = \frac{\delta Z_1 + 2k\delta Z_m}{jnZ_0} . \, I_1^2 \tag{17}$$

where k is the ratio of the currents in the aerials, i.e. I_2/I_1.

By definition δZ_1, the change of self impedance of the unipole, is independent of the second aerial and is given by equations (5) and (10). The change in Z_m will be given approximately by:

$$\delta Z_m = \tfrac{1}{2} n Z_m \frac{\sin^2\theta}{\sin^2\frac{\phi_0}{2}} \left(\tan\frac{\phi_0}{2} - \tan\frac{\theta}{2}\right) \tag{18}$$

Equation (18) assumes that the perturbation changes the amplitude of the field radiated by the unipole but not its v.r.p. This is the same assumption which has already been made in computing the change of input resistance of an isolated unipole and is discussed in the Appendix.

From equations (5), (10), (17) and (18) the square of the current distribution on the unipole is thus identical with that on an isolated unipole in which R_R is replaced by

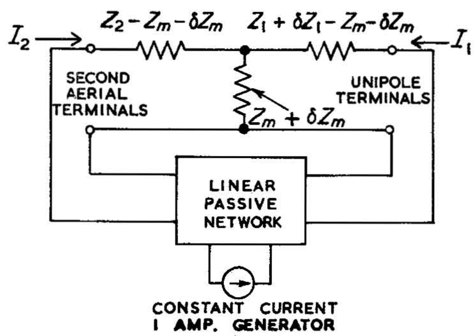

Fig. 4 - Equivalent circuit with series perturbation inserted in unipole

$(R_R + kZ_m)$. Provided that $(R_R + kZ_m)$ is sufficiently small compared with Z_0, the square root can be approximated and the expression for the current is that given in equation (13) with $(R_R + kZ_m)$ substituted for R_R. The mutual impedance will in general be complex and thus both the in-phase and quadrature components of the total aerial current will be modified. In the case of the $\lambda/4$ unipole the current distribution for a unit input current to the unipole will be:

$$I(\theta) = \sin\theta \left[1 - j\,\frac{(R_R + kZ_m)(1 - \tan\frac{\theta}{2})}{Z_0}\right] \tag{19}$$

5. CURRENT DISTRIBUTION ON A HALF-WAVE UNIPOLE

The case of a $\lambda/2$ unipole is of considerable interest since it forms the basis of the m.f. "antifading" type of transmitting aerial. The perturbation method outlined in this report cannot be applied directly to this case since the sinusoidal first approximation of current is zero at the drive point and a large error would occur in the calculation of the "feed" current. The error occurs because the necessary condition that the secondary current must be small compared with the sinusoidal primary current, is not satisfied. The current distribution already determined for a $\lambda/4$ unipole can, however, be used to obtain the feed current distribution for a $\lambda/2$ unipole.

Fig. 5 shows the case considered and the initial sinusoidal approximation of current; the "loop" current is assumed to be unity. The power radiated can be determined by integration of the distant field and since the loop current is I amp it will be numerically equal to R_L, the radiation resistance referred to the current loop. On the initial assumption that the aerial behaves as a lossless transmission line, the input voltage must be Z_0 volts and since the input power is equal to R_L the input current must be R_L/Z_0. It should be noted that the input current must be advanced in plane by $\pi/2$ radians relative to the loop current and an approximation to the true current, taking into account radiation, is therefore known at two points on the aerial, the mid point (the current loop) and the drive point; this is illustrated in Fig. 6. The current distribution will not be changed if an infinite impedance generator giving unit current is inserted at the current loop as illustrated in Fig. 7. The current on the unipole can thus be found by the superposition of

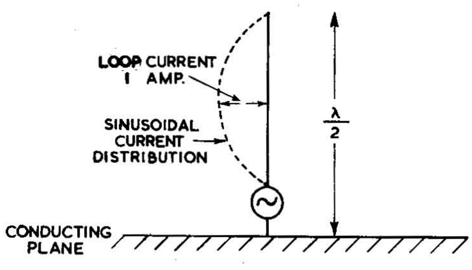

Fig. 5 - Half-wavelength unipole

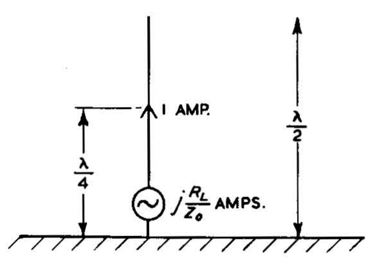

Fig. 6 - Input current of half-wavelength unipole

the currents obtaining in the two arrangements of Figs. 8(a) and (b). In computing the current distribution of arrangement 8(a) equation (19) is used, the values of Z_m being determined from the formulae for mutual impedance between half-wave dipoles given by Carter[12] (the value of mutual impedance thus derived is $Z_m = 26{\cdot}4 + j20{\cdot}2$ ohms). In computing the current distribution for Fig. 8(b) the current on the section marked A is ignored since it will be small. The resulting current distribution for a $\lambda/2$ unipole of characteristic impedance of 250 ohms is shown in Fig. 9 together with the result of a calculation made by G.D. Monteath based on Böhm's theory;[2] the agreement is seen to be good. Figs. 10 and 11 show comparisons of measured[13] and calculated results for the amplitude and phase of the current on a $\lambda/2$ unipole and once again the agreement is seen to be good.

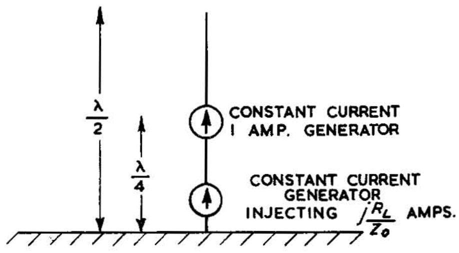

Fig. 7 - Arrangement of constant current generators

6. RESONANT LENGTH OF DIPOLES AND UNIPOLES

It has been mentioned in the Introduction that cylindrical unipole and dipole aerials resonate (i.e. have zero input reactance or susceptance) for lengths less than multiples of $\lambda/4$ in free space and that this effect is due to the in-phase component of the secondary current. The perturbation method of determining the feed current depends on a sinusoidal first approximation to the current distribution; this need not necessarily assume a velocity of propagation equal to that of light. The initial sinusoidal current distribution may be conceived to correspond to a reduced velocity of propagation and this will clearly result in a more accurate prediction of the total current; the calculated curves in Figs. 10 and 11 have been derived in this way. The perturbation method does not seem to lead readily to an estimate of

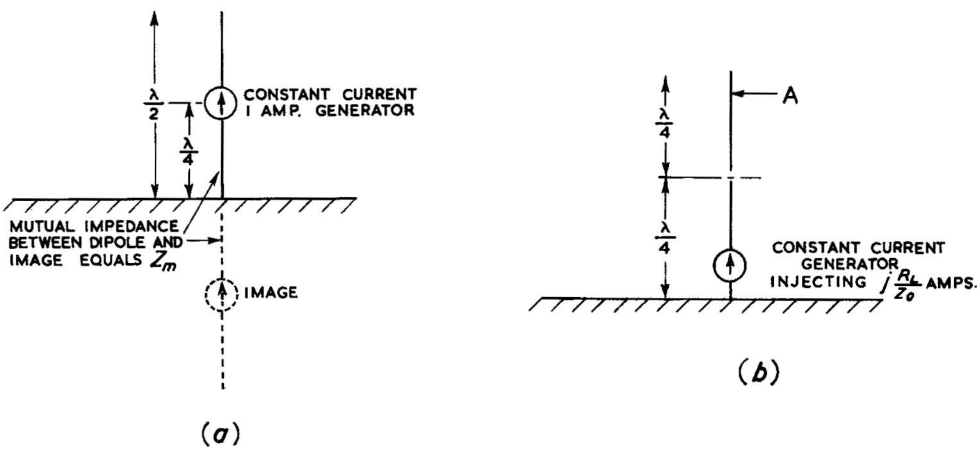

Fig. 8 - Superposition of generators for half-wavelength unipole

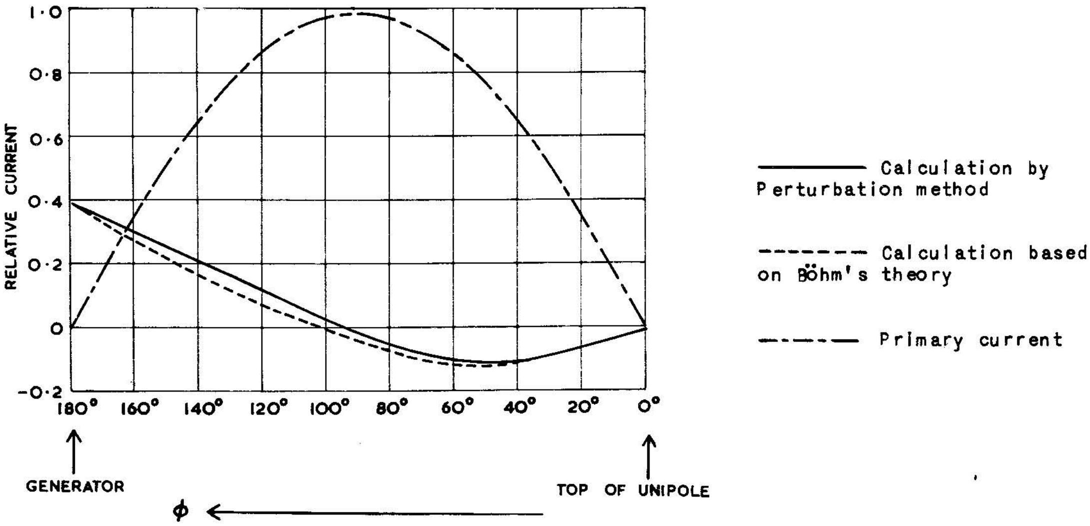

Fig. 9 - Comparison of calculated current on half-wavelength unipole

reduction of velocity although a figure could be deduced by consideration of a new approximate transmission line including the effect of "feed" current. However, predictions of the reduction of velocity along the aerial given by other theories do not agree with measured results and measured figures[1] have been used in the calculations for Figs. 10 and 11.

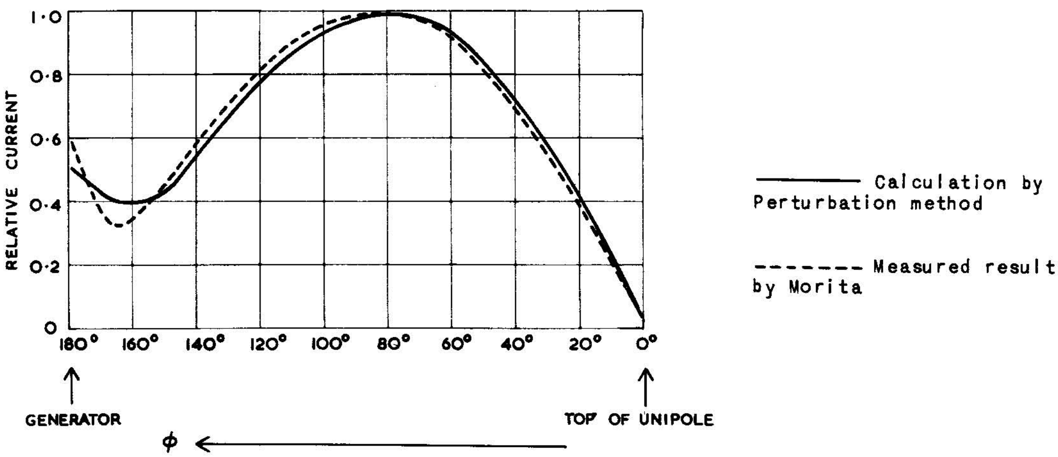

Fig. 10 - Comparison of measured and calculated total current on half-wavelength unipole

7. CONCLUSIONS

A simple method is given of obtaining a correction to the first order assumption of a sinusoidally distributed current on a cylindrical dipole. The correction term derived is in quadrature with the sinusoidal current and corresponds

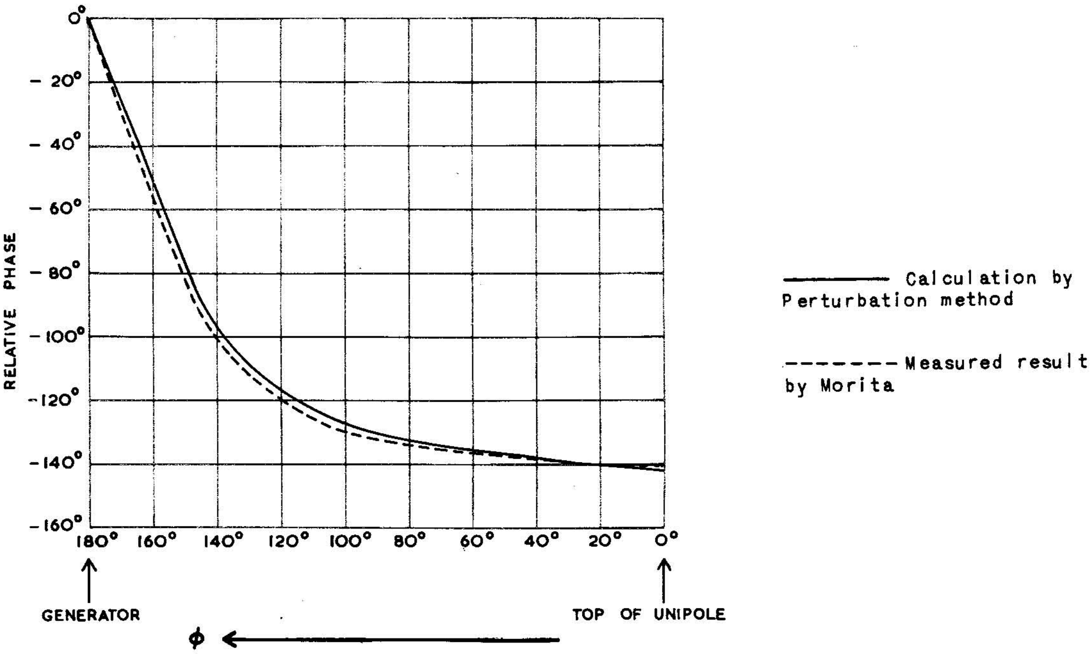

Fig. 11 - Comparison of measured and calculated phase characteristic for half-wavelength unipole

to the "feed" current, i.e. the current component associated with radiation of power. The method is applicable to unipoles of any length up to $\lambda/4$ (or to the corresponding balanced dipoles) and can be extended by superposition to longer aerials. The method also enables the current distribution to be assessed in the presence of other aerials.

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APPENDIX

The Power Gain of a Perturbed Aerial

In this report, the change in radiation resistance due to perturbation of a unipole by a small series reactance is found by assuming that the power gain of the unipole is unchanged, i.e. that the radiation pattern is unchanged by the presence of the perturbation. This assumption is an approximation and the power gain will in fact be modified slightly but the error will be negligible for very short unipoles where the v.r.p. is virtually independent of the current distribution. The error increases as the unipole is made longer and the worst case considered is that of a $\lambda/4$ unipole. In this Appendix it is shown that even with a $\lambda/4$ unipole the error in the change of radiation resistance is less than 3% and results in a negligible error of the calculated value of the current; the error for lower unipole heights will be correspondingly smaller.

On the initial assumption of a lossless transmission line the current distribution on a perturbed $\lambda/4$ unipole is given by equation (7) which can be re-written as:

$$I(\phi) = A \sin\phi + B \sin(\phi - \theta) \tag{20}$$

where $A = 1 + n \sin\theta \cos\theta$

and $B = 0$ for $0 \leqslant \phi \leqslant \theta$

$B = -n \sin\theta$ for $\theta \leqslant \phi \leqslant \pi/2$

Thus the current consists of two sinusoidal terms, one extending along the whole length of the unipole and the other extending from the point of perturbation back to the input terminals. Because the value of the perturbation jnZ_0 is small compared with Z_0 we have $n \ll 1$ and the second term is much smaller than the first.

The current on an unperturbed unipole is $\sin\phi$ and with a suitable choice of units the radiated power is:

$$P_1 = \int_0^{\pi/2} \frac{\cos^2(\pi/2 \sin\psi)}{\cos\psi}\, d\psi = 0{\cdot}6093 \tag{21}$$

where ψ is the angle measured from the horizontal and the solution of the integral is obtained from tabulated values of the cosine integral.

With the same choice of units, the radiated power when the unipole is perturbed is:

$$P_2 = (1 + 2n \sin\theta \cos\theta) \int_0^{\pi/2} \frac{\cos^2(\pi/2 \sin\psi)}{\cos\psi}\, d\psi - 2n \sin\theta (1 - \sin\theta) \int_0^{\pi/2} \cos(\pi/2 \sin\psi) \,.\, F(\psi)\, d\psi \tag{22}$$

where $F(\psi)$ is the v.r.p. of the current distribution term $B \sin(\phi - \theta)$ in equation (20) normalized to unity in the horizontal plane. $F(\psi)$ must lie between the v.r.p.s of a $\lambda/4$ unipole and a very short unipole, i.e. between $\cos(\pi/2 \sin\psi)/\cos\psi$ and $\cos\psi$. Hence the value of the second integral in equation (22) must lie between

$$\int_0^{\pi/2} \frac{\cos^2(\pi/2 \sin\psi)}{\cos\psi} d\psi$$

and

$$\int_0^{\pi/2} \cos(\pi/2 \sin\psi)\cos\psi . d\psi,$$

that is between 0·6093 and 0·6366. The solution of the first of these two integrals was given in equation (21) and the second is readily integrable directly.

If the second integral in equation (22) is assumed to be equal to 0·6093 the power P_2 is that derived on the assumption of unchanged gain when the unipole is perturbed. The error in assessing the radiated power is therefore given by:

$$2n\eta \sin\theta \, (1 - \sin\theta) \tag{23}$$

where η is a number which is equal to or less than 0·6366 − 0·6093 = 0·0273.

From equation (10) with $\phi_0 = \pi/2$ the computed change in radiated power due to the presence of the perturbation is (on the assumption of unchanged gain):

$$2n \sin^2\theta(1 - \tan\theta/2)P_1 \tag{24}$$

and hence the percentage error in the change in radiated power is:

$$= \frac{100\eta}{0 \cdot 6093} \cdot \frac{(1 - \sin\theta)}{[\sqrt{2} \sin(\theta + \pi/4) - 1]} \tag{25}$$

Equation (25) is zero when $\theta = \pi/2$ and also when $\theta = 0$ (because $\eta = 0$ when $\theta = 0$) and the maximum error may thus be expected in the region of $\theta = \pi/4$. If the upper limit of 0·0273 is substituted for η, and $\theta = \pi/4$, equation (25) shows that the maximum error in the assessment of radiated power is less than:

$$\frac{100 \,.\, 0 \cdot 0273}{0 \cdot 6093} \cdot \frac{(1 - 0 \cdot 7071)}{(1 \cdot 4142 - 1)} = 3 \cdot 2\%$$

From equations (11) and (12) the error in the secondary current found by the perturbation method is therefore less than 1½% which is negligible.

BRH

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